IMAC

Institut Universitari de Matemàtiques i Aplicacions de Castelló
Universitat Jaume I

Workshop on hyperbolic and shallow water equations

Monday March 30, 2015

Numerical simulation of a climate energy balance model with continents distribution.

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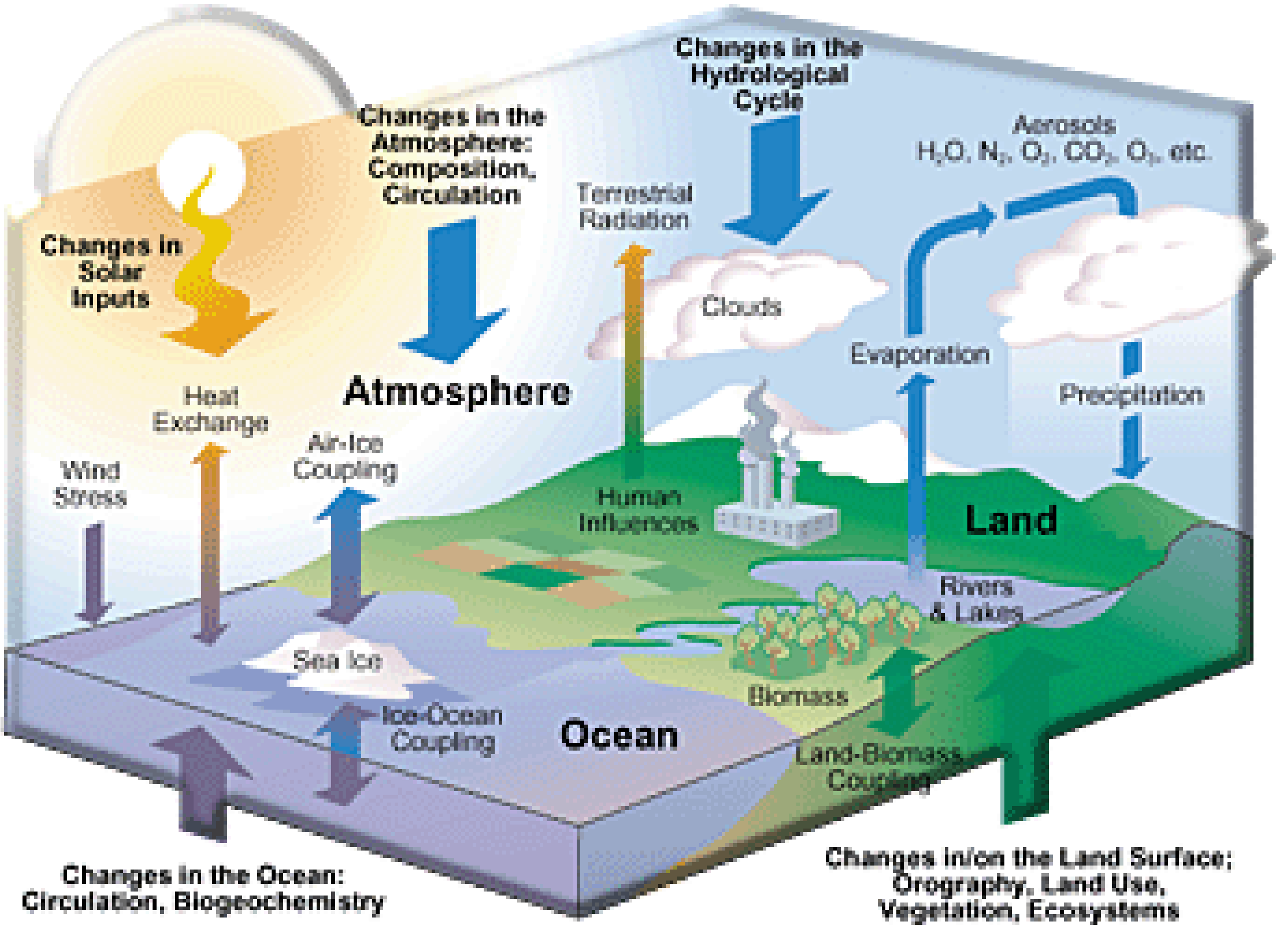
POLITÉCNICA
"Ingeniamos el futuro"

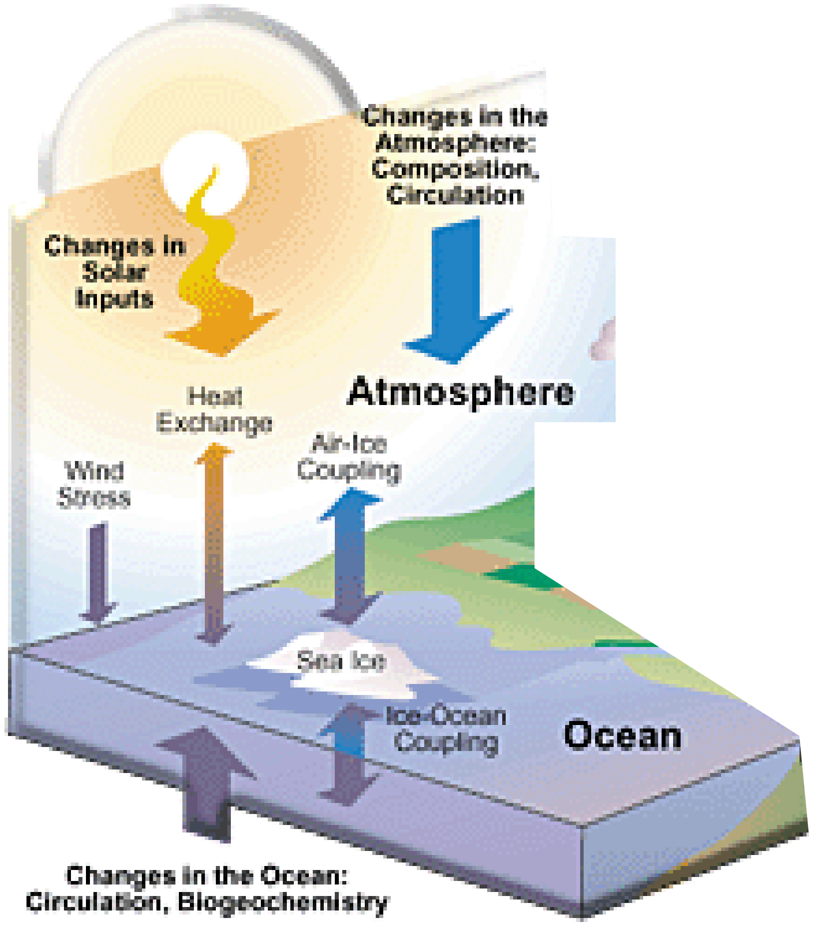
Physical problem

Some processes involved in global climate models:

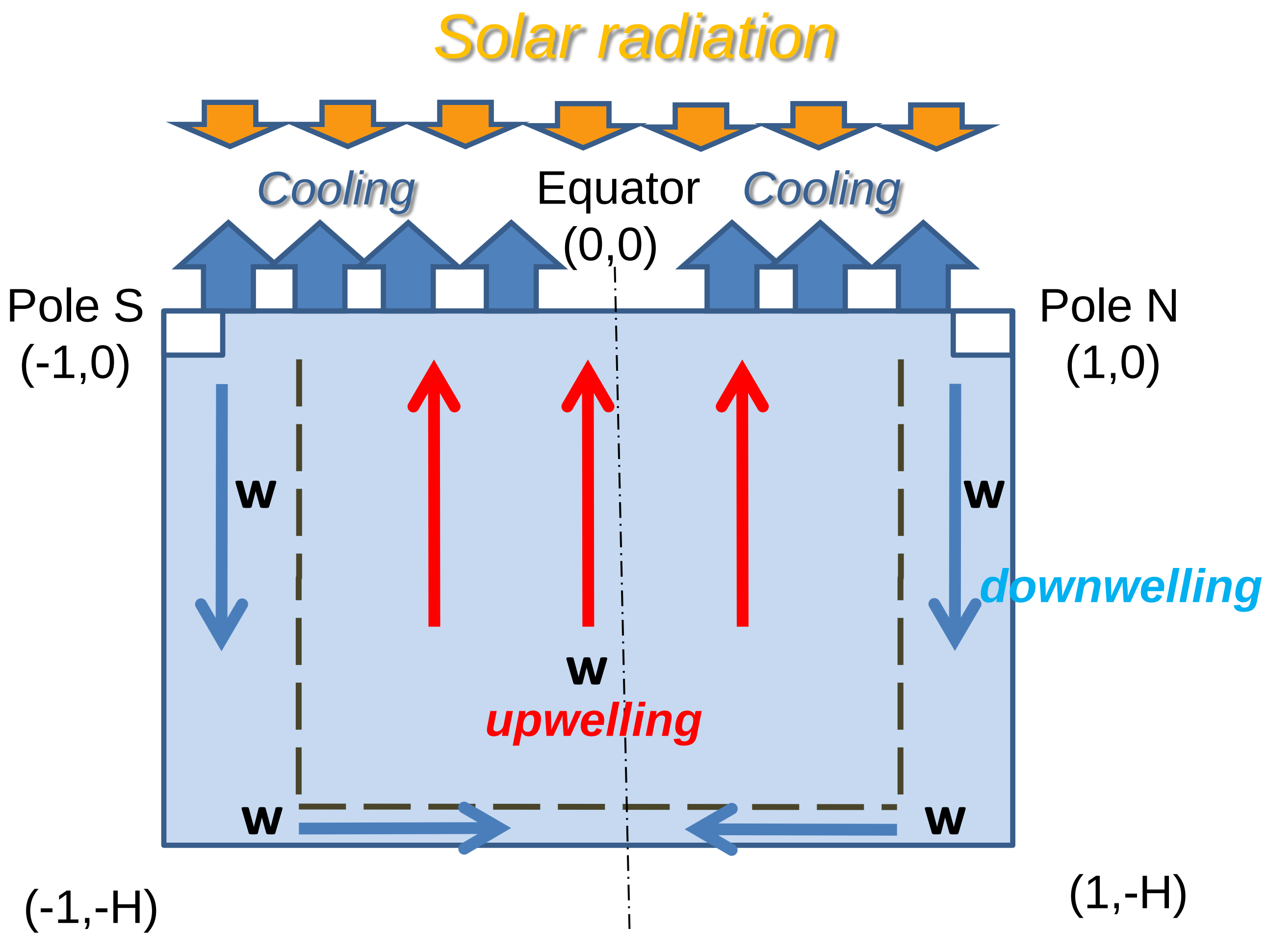


Physical problem

Some processes involved in global climate models:



Physical problem



Mathematical model

THE MODEL (Based on Watts-Morantine [1990])

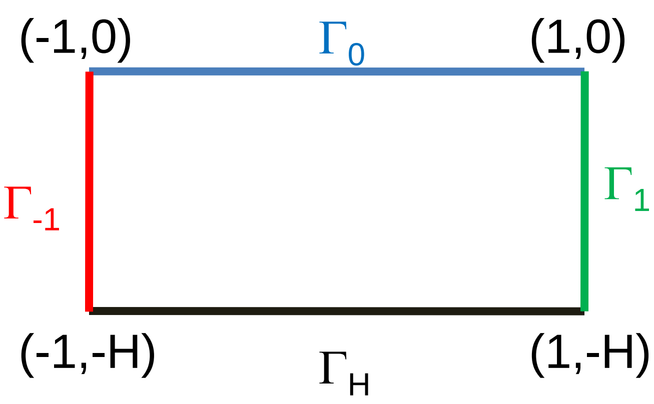
- The model represents the evolution of temperature within an ocean of depth H.
- Spatial variables (x,z): $x = sin$(latitude) and $-z$ (depth).
- Spatial domain $\Omega = (-1,1)\times(0,-H)$
- Boundary: $\Gamma = \Gamma_H \cup \Gamma_0 \cup \Gamma_1 \cup \Gamma_{-1}$

$$\Gamma_H = \{(x,z)\in\bar{\Omega}: \quad z=-H\}$$

$$\Gamma_0 = \{(x,z)\in\bar{\Omega}: \quad z=0\}$$

$$\Gamma_{-1} = \{(x,z)\in\bar{\Omega}: \quad x=-1\}$$

$$\Gamma_1 = \{(x,z)\in\bar{\Omega}: \quad x=1\}$$



- The model considers the average temperature over each parallel as the unknown.

Mathematical model

The governing equation for the ocean interior is a heat equation with advective transport (DOM)

$$U_t - \left(\frac{K_H}{R^2}(1-x^2)U_x\right)_x - K_V U_{zz} + \omega U_z = 0 \quad \text{in } (0,T)\times\Omega_i, \qquad i = 1 \ldots N.$$

U: temperature,
ω: vertical velocity,
K_V: vertical diffusivity,
K_H: horizontal diffusivity,
R: radius of the Earth.

Mathematical model

Boundary condition for ocean bottom

$$\omega x U_x + K_v U_z = 0, \quad on \quad \Gamma_H \times (0,T)$$

Boundary condition for upper boundary: Energy Balance Model (EBM)

$$Du_t - \frac{DK_{H_0}}{R^2}\left((1-x^2)^{\frac{p}{2}} \left| u_x \right|^{p-2} u_x \right)_x + Bu + C + K_V \frac{\partial U}{\partial z} + \omega x u_x \in \frac{1}{\rho c} QS(x)\beta(u)$$
$$on \quad \Gamma_0 \times [0,T]$$

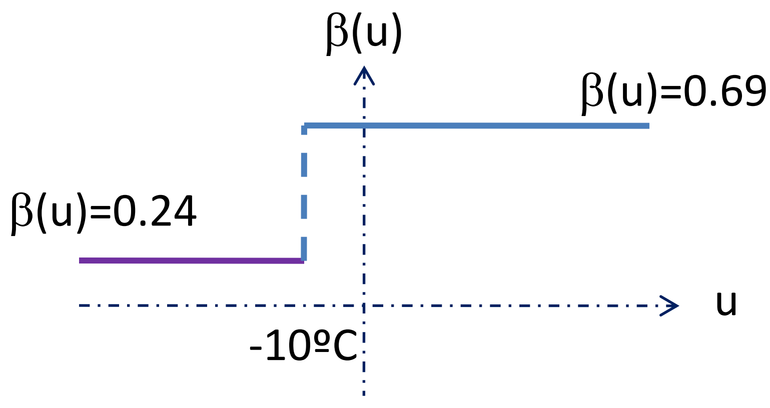
u: temperature,
ω: velocity,
K_v: vertical diffusivity,
K_{H0}: horizontal diffusivity,
R: radius of the Earth.

D: thickness mixed layer,
ρ: density,
c: specific heat coeff.,
β(u): coalbedo,
Q: solar constant,
Bu+C: cooling term,
S(x): insolation.

Mathematical model

1) We consider the case p=3 (Stone, 1972)

2) We use the coalbedo, $\beta(u)$, (Budyko model)



Mathematical model

$$
\left\{
\begin{array}{l}
U_t - \left(\dfrac{K_H}{R^2}(1-x^2)U_x\right)_x - K_V U_{zz} + \omega U_z = 0 \quad \text{in } \Omega \times (0,T), \\[2ex]
\omega x U_x + K_V U_z = 0 \quad \text{in } \Gamma_H \times (0,T), \\[2ex]
Du_t - \dfrac{DK_{H_0}}{R^2}\left((1-x^2)^{3/2}|u_x|u_x\right)_x + K_V \dfrac{\partial U}{\partial z} + \omega x u_x + Bu + C \in \dfrac{1}{\rho c} QS(x)\beta(u) \\
\text{on } \Gamma_0 \times (0,T), \\[2ex]
U_x = 0, \quad on \quad \Gamma_{-1} \times [0,T] \quad , \\[2ex]
U_x = 0, \quad in \quad \Gamma_1 \times [0,T] \quad , \\[2ex]
U(x,z,0) = U_0(x,z), \quad in \quad \Omega \quad , \\
u(x,0) = u_0(x), \qquad in \quad \Gamma_0 \quad .
\end{array}
\right.
$$

Final system

Mathematical model

$$\frac{K_v}{D}\frac{\partial U}{\partial z}$$

Represents the coupling atmosphere-ocean in the sense of analyzing the influence of the ocean temperature in the atmosphere.
In this work we shall show results with and without this term.

Some references about global climate EBM models with or without deep ocean effect :

- **Watts&Morantine (1990),**
- **Xu (1990),**
- **Hetzer (1990),**
- **Kim, North & Huang (1992),**
- **Díaz (1993),**
- **Schmidt (1994),**
- **Díaz-Hernández-Tello (1997),**
- **Arcoya-Díaz-Tello (1998),**
- **Hetzer (2000),**
- **Díaz-Tello (2007),**
- **Bermejo et al (2008),**
- **Hidalgo-Tello (2011,2013,2014,2015),**
- **…**

Numerical approximation

We rewrite this problem as advection-reaction-diffusion equations, both for the upper boundary EBM and for the DOM.

EBM:

$$u_t - \Big(f\big(x, u(x,t), u_x(x,t)\big)\Big)_x = \sigma(x, u(x,t), \frac{\partial U}{\partial z}(x,0,t))$$

with the flux:

$$f\big(x, u(x,t), u_x(x,t)\big) := \frac{K_{H_0}}{R^2}(1-x^2)^{3/2}\left|u_x(x,t)\right| u_x(x,t) - \frac{w}{D} x u(x,t)$$

and the source term:

$$\sigma(x, u, \frac{\partial U}{\partial n}) := \frac{1}{D}(-C + \frac{Q}{\rho c} S(x)\beta(u) + (\omega + x\omega_x - B)u(x,t) - K_V \frac{\partial U}{\partial z})$$

Numerical approximation

DOM:

$$U(x,z,t)_t - (F(x,U_x(x,z,t)))_x - (G(U(x,z,t),U_z(x,z,t)))_z = \Xi(x,U(x,z,t)),$$

with the fluxes :

$$F\left(x,U_x(x,z,t)\right) := \frac{K_H}{R^2}\left(1-x^2\right)U_x(x,z,t),$$
$$G(U(x,z,t),U_z(x,z,t)) := K_V U_z(x,z,t) - wU(x,z,t),$$

and the source term:

$$\Xi(x,U(x,z,t)) := \omega_z U(x,z,t).$$

<u>Numerical approach</u>: finite volume method with Weighted Essentially Non-Oscillatory (WENO) reconstruction in space and third-order Runge-Kutta TVD for time integration.

For each time step, we compute a numerical solution of the EBM model equation for each cell u_i^{n+1}

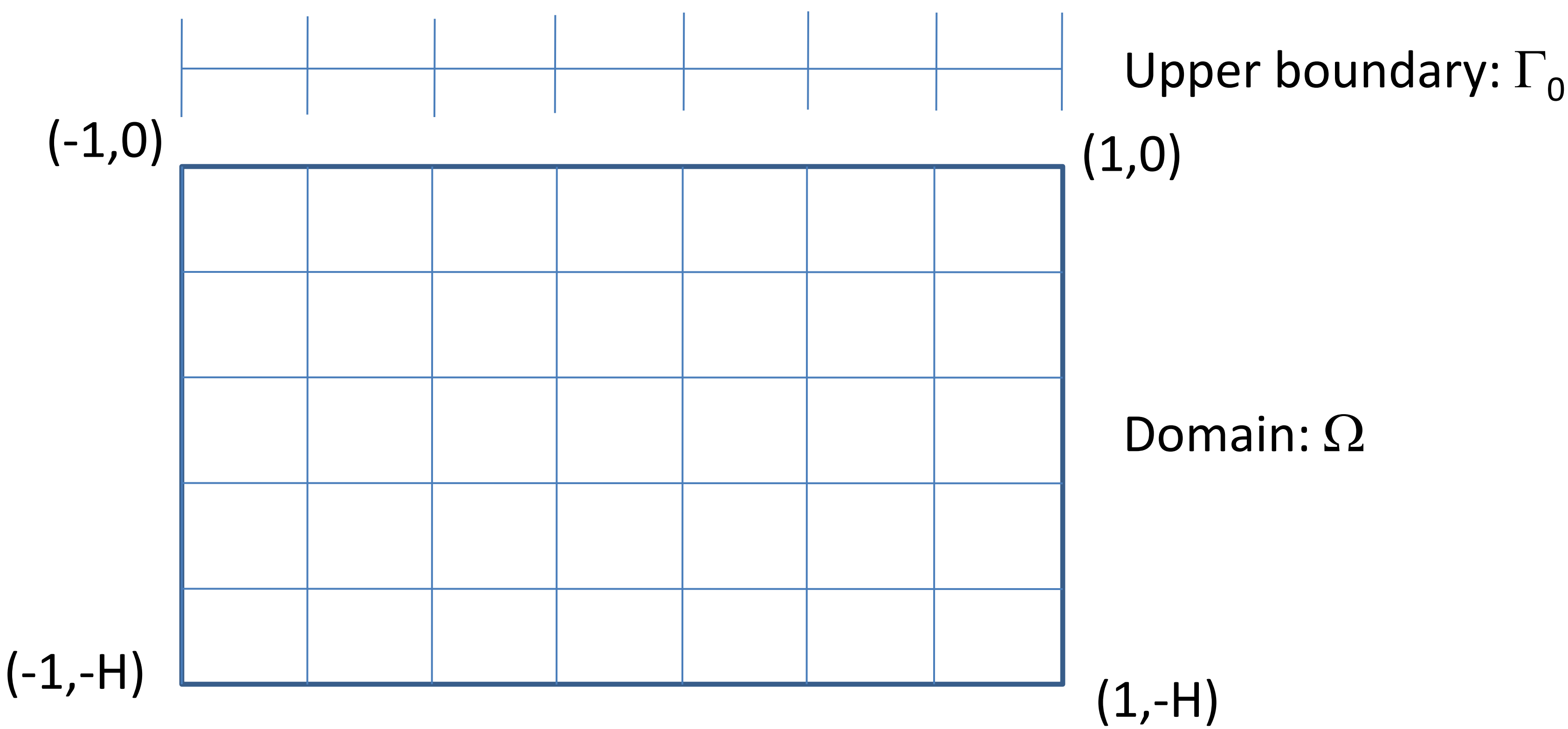
$$Du_t - \frac{DK_{H_0}}{R^2}\left((1-x^2)^{\frac{p}{2}} \left|u_x\right|^{p-2} u_x \right)_x + Bu + C + K_V \frac{\partial U}{\partial z} + \omega x u_x \in QS(x)\beta(u)$$

$$on \quad \Gamma_0 \times \left[0,T\right]$$

then we use u_i^{n+1} as a Dirichlet boundary condition for the DOM to obtain $U_{i,j}^{n+1}$

$$U_t - \left(\frac{K_H}{R^2}(1-x^2)U_x\right)_x - K_V U_{zz} + \omega U_z = 0 \quad \text{in } \Omega \times (0,T)$$

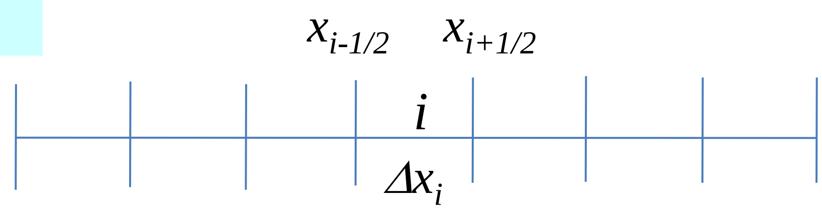
The finite volume framework



The finite volume framework

We integrate the equation dividing by the length of the control volume to obtain the following ordinary differential equation (ODE)

$$\frac{du_i(t)}{dt} = \frac{1}{\Delta x_i}\left(f_{i+1/2} - f_{i-1/2} \right) + \sigma_i(t) \equiv l_i(u(t)),$$



where

$$u_i(t) = \frac{1}{\Delta x_i}\int_{x_{i-1/2}}^{x_{i+1/2}} u\left(x,t\right)dx$$ integral average of the unknown,

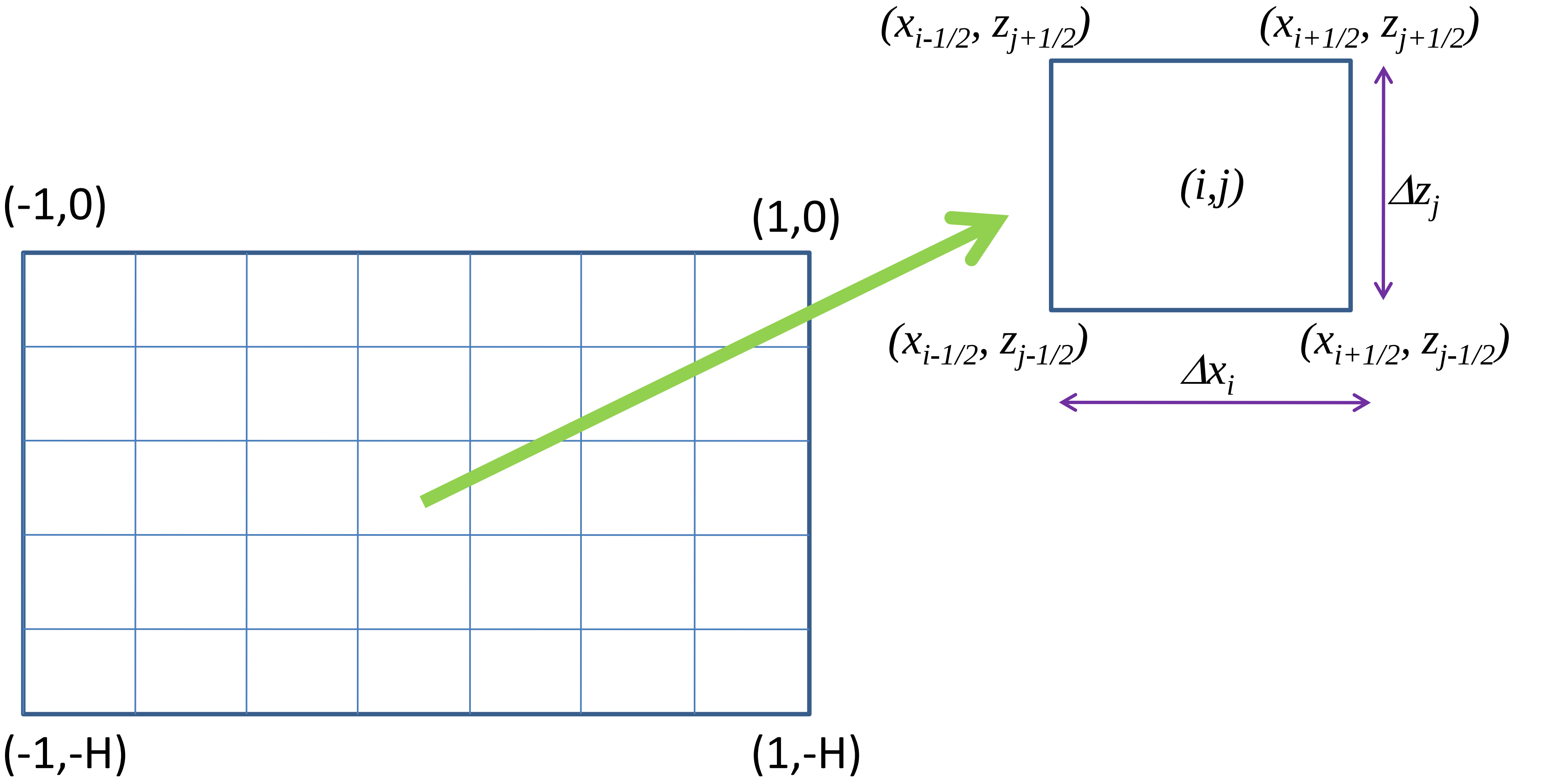
$$f_{i+1/2} = f\left(\mathrm{x}, u\left(x_{i+1/2}, t\right), u_x\left(x_{i+1/2}, t\right)\right)$$ right intercell numerical flux,

$$\sigma_i(t) = \frac{1}{\Delta x_i}\int_{x_{i-1/2}}^{x_{i+1/2}} \sigma(x, u, \frac{\partial U}{\partial z})dx$$ integral average of the source term.

The finite volume framework

We discretize the 2D domain [-1,1] $\times$[0,-H] in N_xN_z control volumes of area $\Delta x_i \times \Delta z_j$

$\Delta x_i = x_{i+1/2} - x_{i-1/2}$, $\Delta z_j = z_{j+1/2} - z_{j-1/2}$



The finite volume framework

We integrate the equation dividing by the area of the control volume to obtain the following ordinary differential equation (ODE)

$$\frac{dU_{i,j}}{dt} = \frac{1}{\Delta x_i}\left(F_{i+1/2,j} - F_{i-1/2,j}\right) + \frac{1}{\Delta z_j}\left(G_{i,j+1/2} - G_{i,j-1/2}\right) + \Gamma_{i,j} \equiv L_{i,j}$$

where

$$U_{i,j} = \frac{1}{\Delta x_i \Delta z_j}\int_{z_{j-1/2}}^{z_{j+1/2}}\left(\int_{x_{i-1/2}}^{x_{i+1/2}} U(x,z,t)dx\right)dz$$ integral average of the unknown,

$$F_{i+1/2,j} = \frac{1}{\Delta z_j}\int_{z_{j-1/2}}^{z_{j+1/2}} F\left(x_{i+1/2}, U_x\left(x_{i+1/2}, z, t\right)\right)dz,$$ Spatial integral average of intercell fluxes,

$$G_{i,j+1/2} = \frac{1}{\Delta x_i}\int_{x_{i-1/2}}^{x_{i+1/2}} G\left(U\left(x, z_{j+1/2}, t\right), U_z\left(x, z_{j+1/2}, t\right)\right)dx,$$

$$\Gamma_{i,j}(t) = \frac{1}{\Delta x_i \Delta z_j}\int_{z_{j-1/2}}^{z_{j+1/2}}\left(\int_{x_{i-1/2}}^{x_{i+1/2}} \Xi(x, U(x,z,t))dx\right)dz,$$ integral average of the source term.

Runge Kutta TVD

EBM:

$$u^{k,1} = u^n + \Delta t l(u^n), \qquad u^{k,2} = \frac{3}{4}u^n + \frac{1}{4}u^{k,1} + \frac{1}{4}\Delta t l(u^{k,1}),$$

$$u^{n+1} = \frac{1}{3}u^n + \frac{2}{3}u^{k,2} + \frac{2}{3}\Delta t l(u^{k,2}).$$

DOM:

$$U^{k,1} = U^n + \Delta t L(U^n),$$

$$U^{k,2} = \frac{3}{4}U^n + \frac{1}{4}U^{k,1} + \frac{1}{4}\Delta t L\left(U^{k,1}\right),$$

$$U^{n+1} = \frac{1}{3}U^n + \frac{2}{3}U^{k,2} + \frac{2}{3}\Delta t L\left(U^{k,2}\right).$$

WENO reconstruction

EBM

1) For intercell fluxes

For an order of accuracy r we have r candidate stencils each one of them with r cells

$$\{S_{i-r+1}, S_{i-r+2},\ldots,S_i\}, \{S_{i-r+2}, S_{i-r+3},\ldots,S_{i+1}\}, \ldots , \{S_i, S_{i+1},\ldots,S_{i+r-1}\}$$

For each stencil we consider a (r-1)th degree interpolating polynomial

$$p_l(x), \quad l = 0,\cdots,r-1$$

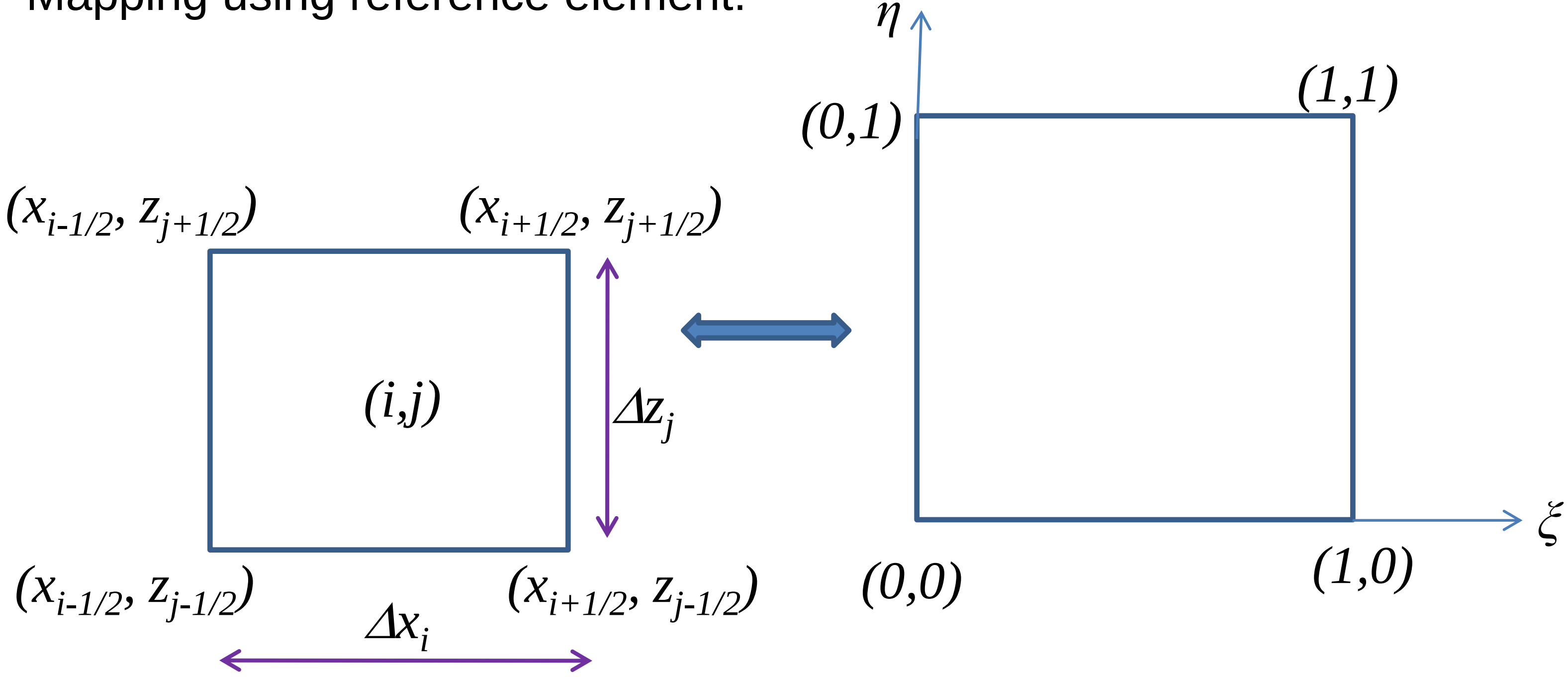
Each one of the polynomials considered must be conservative:

$$\frac{1}{\Delta x_k}\int_{S_k} p_l(x)dx = u_k(t), \qquad 0 \le l \le r-1, \quad 0 \le k \le r-1$$

Remark: In this work we have used r=4. Therefore, the candidate stencils are:

$\{S_{i-3}, S_{i-2}, S_{i-1},S_i\}, \{S_{i-2}, S_{i-1}, S_i,S_{i+1}\},$
$\{S_{i-1},S_i, S_{i+1},S_{i+2}\},\{S_i,S_{i+1}, S_{i+2},S_{i+3}\}.$

Mapping using reference element:



$$x = x_{i-1/2} + \Delta x_i \xi$$

$$z = z_{j-1/2} + \Delta z_j \eta$$

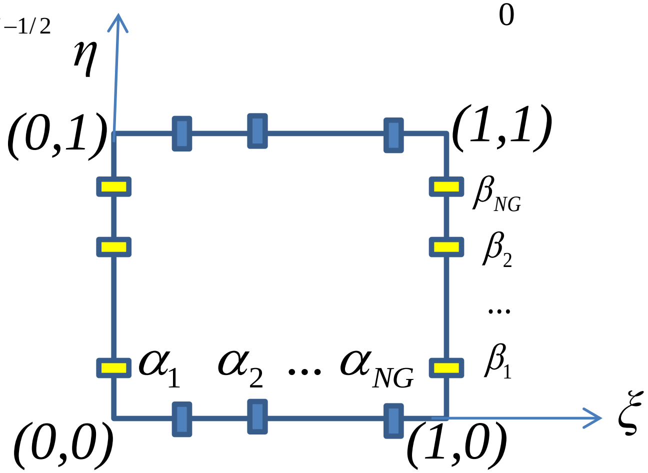
Integrals are approximated using Gaussian numerical quadrature

$$F_{i+1/2,j} = \frac{1}{\Delta z_j}\int_{z_{j-1/2}}^{z_{j+1/2}} F(x_{i+1/2}, z, t^n)dz = \int_0^1 \hat{F}(1,\eta,t^n)d\eta \approx \sum_{k=1}^{NG}\gamma_k^z \hat{F}(1,\beta_k,t^n)$$

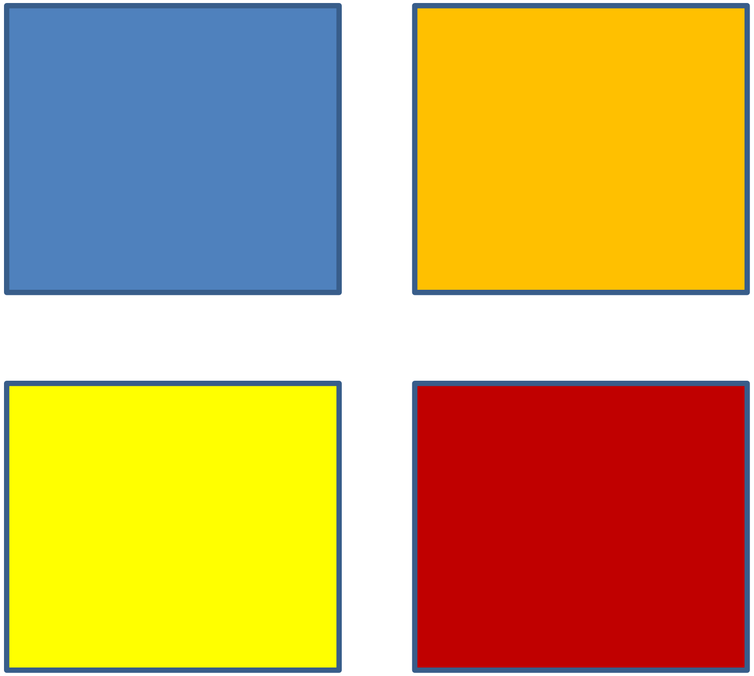
$$F_{i-1/2,j} = \frac{1}{\Delta z_j}\int_{y_{j-1/2}}^{y_{j+1/2}} F(x_{i+1/2}, z, t^n)dz = \int_0^1 \hat{F}(0,\eta,t^n)d\eta \approx \sum_{k=1}^{NG}\gamma_k^z \hat{F}(0,\beta_k,t^n)$$

$$G_{i,j+1/2} = \frac{1}{\Delta x_i}\int_{x_{i-1/2}}^{x_{i+1/2}} G(x, z_{j+1/2}, t^n)dx = \int_0^1 \hat{G}(\xi,1,t^n)d\xi \approx \sum_{k=1}^{NG}\gamma_k^x \hat{G}(\alpha_k,1,t^n)$$

$$G_{i,j-1/2} = \frac{1}{\Delta x_i}\int_{x_{i-1/2}}^{x_{i+1/2}} G(x, z_{j-1/2}, t^n)dx = \int_0^1 \hat{G}(\xi,0,t^n)d\xi \approx \sum_{k=1}^{NG}\gamma_k^x \hat{G}(\alpha_k,0,t^n)$$



We must compute a unique flux at each control volume interface



$$F_{i+1/2,j} = \frac{1}{2}\left(F^{i,j,RIGHT}_{i+1/2,j} + F^{i+1,j,LEFT}_{i+1/2,j}\right)$$

$$G_{i+1/2,j} = \frac{1}{2}\left(G^{i,j,UP}_{i,j+1/2} + G^{i,j+1,DOWN}_{i,j+1/2}\right)$$

<u>REMARKS:</u>

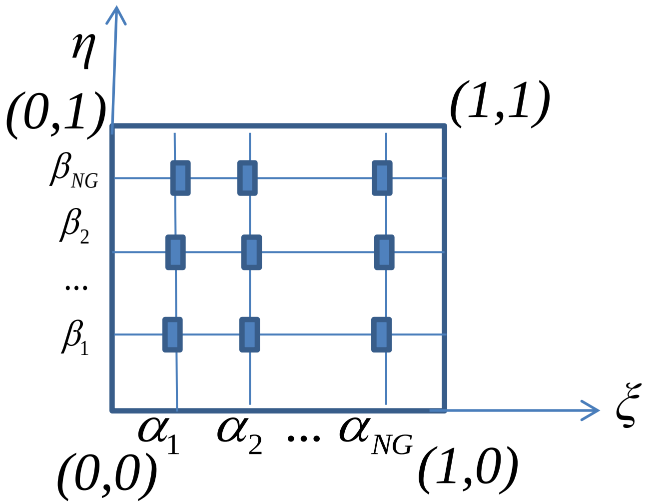
1)When considering advective terms, other types of flux averaging
or Riemann problem solutions must be introduced: Force, Rusanov, Osher, Roe…

2) In the diffusive averaging of the flux a term that accounts for the jump
can be added.

$$\Gamma_{i,j}(t^n) = \frac{1}{\Delta x_i \Delta z_j} \int_{z_{j-1/2}}^{z_{j+1/2}} \left(\int_{x_{i-1/2}}^{x_{i+1/2}} S(x,z,t^n)dx \right) dy =$$

$$= \frac{1}{\Delta x_i \Delta z_j} \int_0^1 \left(\int_0^1 S(x,z,t^n)\Delta x_i d\xi \right) \Delta z_j d\eta \approx$$

$$\approx \sum_{k=1}^{NG} \gamma_k^x \left(\sum_{l=1}^{NG} \gamma_l^z G(\alpha_k, \beta_l, t^n) \right)$$



If we use 2 integration points (NG=2) the values are:

$$\gamma_0^{x,z} = \gamma_1^{x,z} = 1$$

$$\alpha_0 = \beta_0 = -\frac{\sqrt{3}}{3}$$

$$\alpha_1 = \beta_1 = \frac{\sqrt{3}}{3}$$

Dimension-by-Dimension WENO reconstruction

In order to obtain the solution and gradients at Gaussian points we need to perform a reconstruction procedure. This give rise to a piecewise polyonomial function whose restriction to cell T_{ij} is the polynomial:

$$w_{ij}(\xi,\eta,t^n) = \sum_{k=1}^{M+1}\sum_{l=1}^{M+1}\hat{w}_{ij}^{k,l}(t^n)\phi_k(\xi)\phi_l(\eta)$$

which must be conservative:

$$\int_0^1\int_0^1 w_{ij}(\xi,\eta,t^n)d\xi d\eta = Q_i^n \Rightarrow \sum_{k=1}^{M+1}\sum_{l=1}^{M+1}\hat{w}_{ij}^{k,l}(t^n)\int_0^1\int_0^1 \phi_k(\xi)\phi_l(\eta)d\xi d\eta = Q_i^n$$

And its gradient

$$\begin{pmatrix} \dfrac{\partial w_{ij}}{\partial \xi} \\ \dfrac{\partial w_{ij}}{\partial \eta} \end{pmatrix} = \sum_{k=1}^{M+1}\sum_{l=1}^{M+1}\begin{pmatrix} \hat{w}_{ij}^{k,l}(t^n)\phi'_k(\xi)\phi_l(\eta) \\ \hat{w}_{ij}^{k,l}(t^n)\phi_k(\xi)\phi'_l(\eta) \end{pmatrix}$$

Dimension-by-Dimension WENO reconstruction

Substencils for obtaining nonlinear reconstruction operator

$$S_{ij}^{s,x} = \bigcup_{p=i+s-e}^{i+s+e} T_{pj} \qquad \text{and} \qquad S_{ij}^{s,y} = \bigcup_{q=j+s-e}^{j+s+e} T_{qj}$$

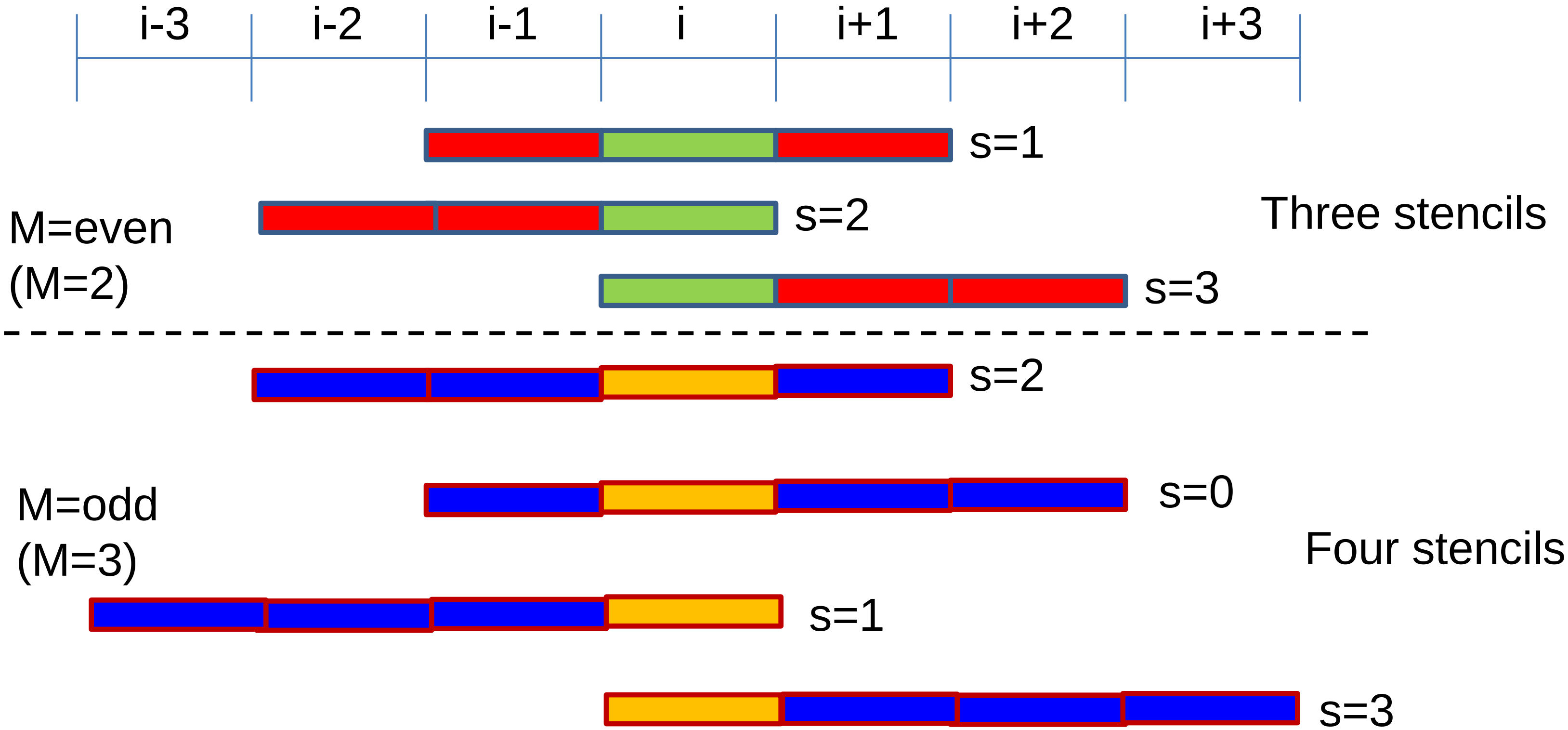
If we consider 3 cells in the stencils: e=1 and s=-e for the left-sided stencil s=0 for the central stencil and s=e for the right-sided stencil.

The total stencils are given by the union of the substencils

$$S_{ij}^{x} = \bigcup_{s} S_{ij}^{s,x} \qquad \text{and} \qquad S_{ij}^{z} = \bigcup_{s} S_{ij}^{s,z}$$

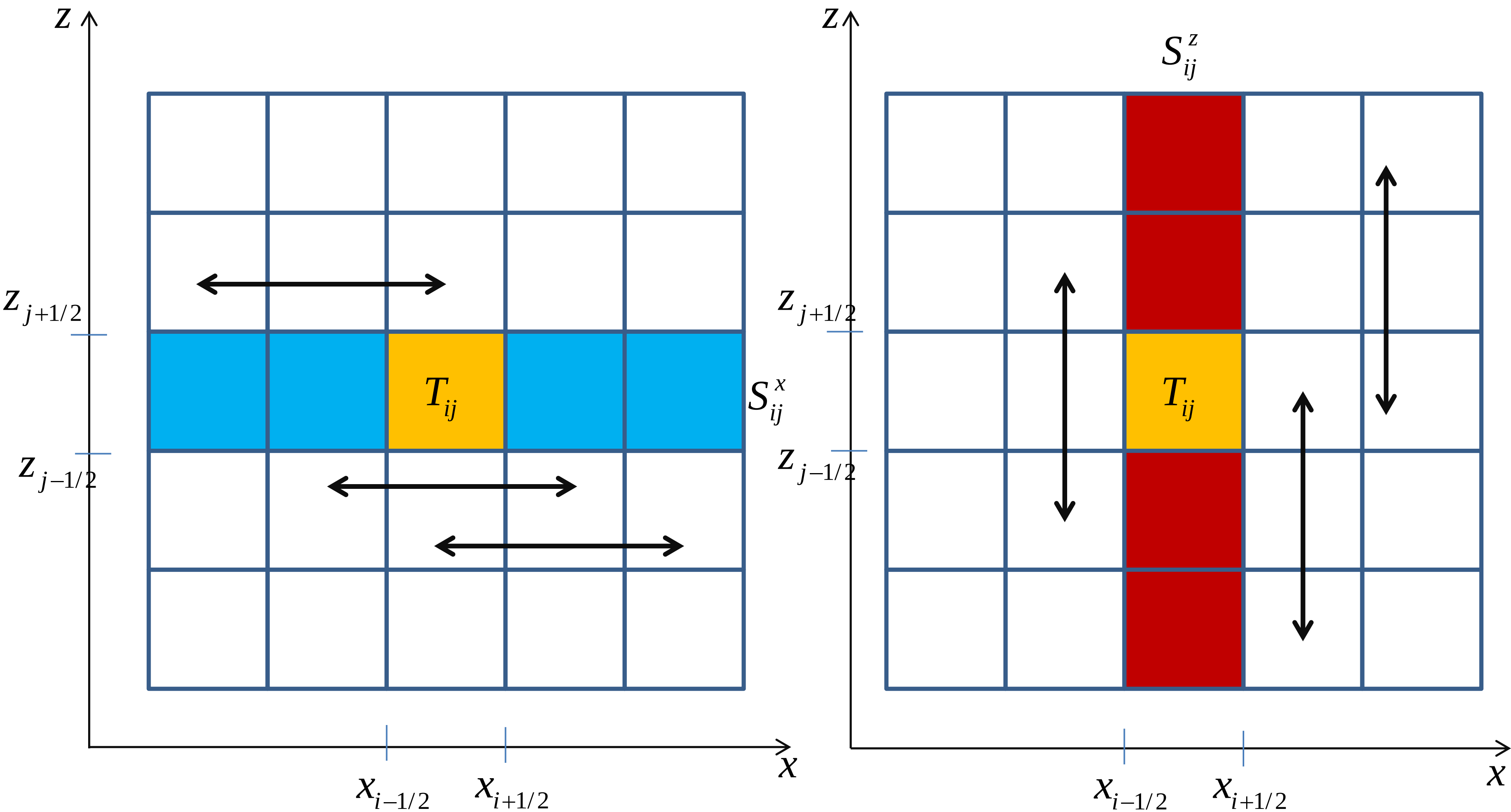
WENO RECONSTRUCTION

$$\mathcal{S}_{ij}^{s,x} = \bigcup_{e=i-L}^{i+R} I_{ej}, \qquad \mathcal{S}_{ij}^{s,z} = \bigcup_{e=j-L}^{j+R} I_{ei}$$



M+1 cells in each stencil

Dimension-by-Dimension WENO reconstruction



Dimension-by-Dimension WENO reconstruction

Let us denote the one-dimensional polynomial WENO reconstruction in x direction as:

$$\hat{w}_l(t^n) = R_x(T_{pj}^n), \quad T_{pj}^n \in S_{ij}^x$$

And the one-dimensional polynomial WENO reconstruction in z direction as:

$$\hat{w}_l(t^n) = R_z(T_{iq}^n), \quad T_{iq}^n \in S_{ij}^z$$

Application of the reconstruction operator in x-direction to the cell averages T_{pj}^n yields the coefficients of the reconstruction polynomial in x-direction.

$$\hat{w}_{ij}^{l,0}(t^n) = R_x(T_{pj}^n), \quad T_{pj}^n \in S_{ij}^x$$

Since the reconstruction operator in z-direction $R_y(T_{iq}^n)$ acts on averages in z-direction, it can be applied to each single coefficient of the reconstruction polynomial in x-direction.

$$\hat{w}_{ij}^{l,m}(t^n) = R_z(\hat{w}_{iq}^{l,0}(t^n)), \qquad \forall 0 \le l \le NG, \quad T_{iq}^n \in S_{ij}^z$$

Dimension-by-Dimension WENO reconstruction

In this way we obtain all the necessary coefficients of the 2D tensor-product reconstruction polynomial

$$w_{ij}(\xi,\eta,t^n) = \sum_{k=1}^{M+1}\sum_{l=1}^{M+1} \hat{w}_{ij}^{k,l}(t^n)\phi_k(\xi)\phi_l(\eta)$$

<u>Remarks</u>:
This way to proceed is different to the classical point-wise WENO approach, since entire polynomials are obtained instead of piecewise values (as in the original WENO Of Jiang and Shu).

Some references of polynomial WENO reconstruction:

*“**High order space-time adaptive ADER-WENO finite volume schemes for non-conservative hyperbolic systems**“, M. Dumbser, A. Hidalgo, O. Zanotti Computer Methods in Applied Mechanics and Engineering, 268, 359-387 (2014)*

*“**ADER-WENO Finite Volume Schemes with Space-Time Adaptive Mesh Refinement**”. M. Dumbser, O. Zanotti, A. Hidalgo, D. Balsara. Journal of Computational Physics, 248, 257-286 (2013)*

Dimension-by-Dimension WENO reconstruction

$$\hat{u}\left(1,\beta_k,t^n\right)=\sum_{k=1}^{M+1}\omega_k w_{ij}(1,\beta_k,t^n);\qquad u\left(0,\beta_k,t^n\right)=\sum_{k=1}^{M+1}\omega_k w_{ij}(0,\beta_k,t^n)$$

$$\hat{u}\left(\alpha_k,1,t^n\right)=\sum_{k=1}^{M+1}\omega_k w_{ij}\left(\alpha_k,1,t^n\right);\qquad u\left(\alpha_k,0,t^n\right)=\sum_{k=1}^{M+1}\omega_k w_{ij}\left(\alpha_k,0,t^n\right)$$

$$\nabla\hat{u}\left(1,\beta_k,t^n\right)=\sum_{k=0}^{r-1}\omega_k\nabla w_{ij}(1,\beta_k,t^n);\qquad \nabla\hat{u}\left(0,\beta_k,t^n\right)=\sum_{k=0}^{r-1}\omega_k\nabla w_{ij}(0,\beta_k,t^n)$$

$$\nabla\hat{u}\left(\alpha_k,1,t^n\right)=\sum_{k=0}^{r-1}\omega_k\nabla w_{ij}\left(\alpha_k,1,t^n\right);\qquad \nabla\hat{u}\left(\alpha_k,0,t^n\right)=\sum_{k=0}^{r-1}\omega_k\nabla w_{ij}\left(\alpha_k,0,t^n\right)$$

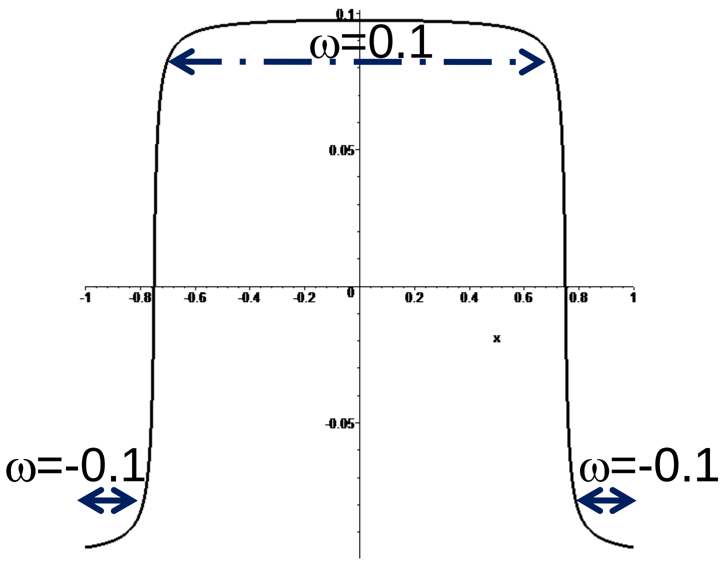
Numerical example without latent heat: γ(U)=U

Physical parameters:

Parameter	Scaled Value
K_H	0.049
K_{H0}	0.555×10^{-3}
K_V	0.0125
C, B	190, 2
c, ρ	1, 1
Q	340
D	60

$$S = 1 - \frac{1}{2}P_2(x)$$

$$\omega(x,z) = W(x) = \frac{10(x+0.75)(x-0.75)}{(0.1+10|x+0.75|)(0.1+10|x-0.75|)}$$



Space and time discretization:

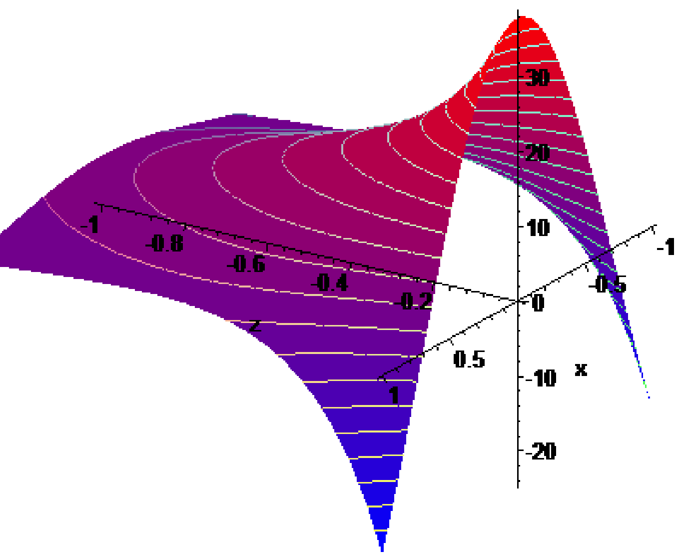
$$\Delta x = 2/60; \qquad \Delta z = 1/60$$

$$\Delta t = \min\left(\alpha\Delta x^2\left(\left(1-x^2\right)KH\right)^{-1}, \alpha\Delta z^2\left(KV\right)^{-1}, \alpha\Delta x^2\left(\left(1-x^2\right)KH0\left|\frac{du}{dx}\right|\right)^{-1}\right), (\alpha = 0.3)$$

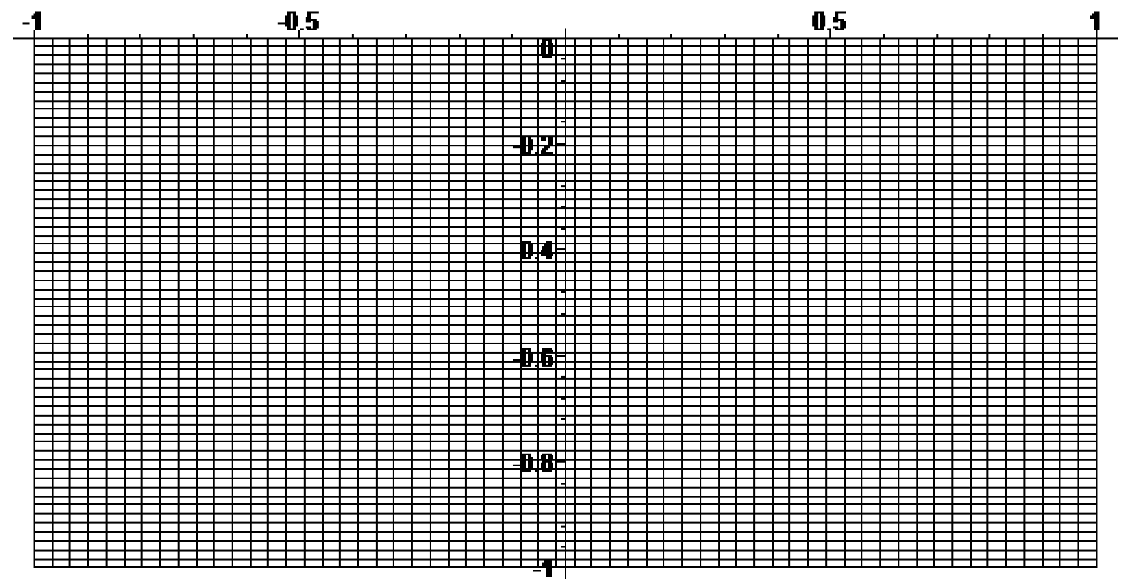
Numerical example

Initial condition:

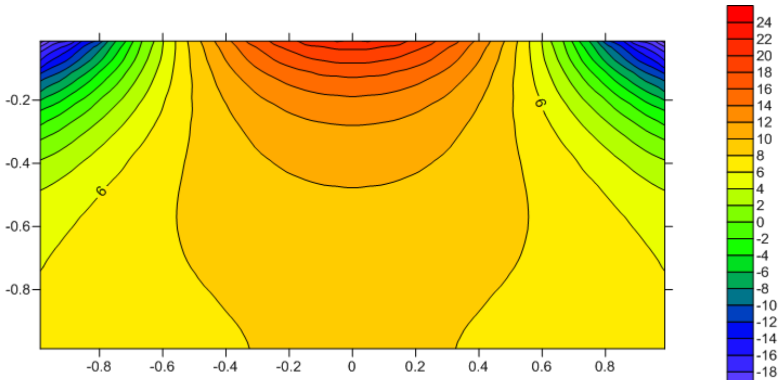
$$U(x,z,0) = 18e^{-x^2-z^2} + 80e^{-x^2} - 60$$



Spatial mesh used (60 x 60 cells)

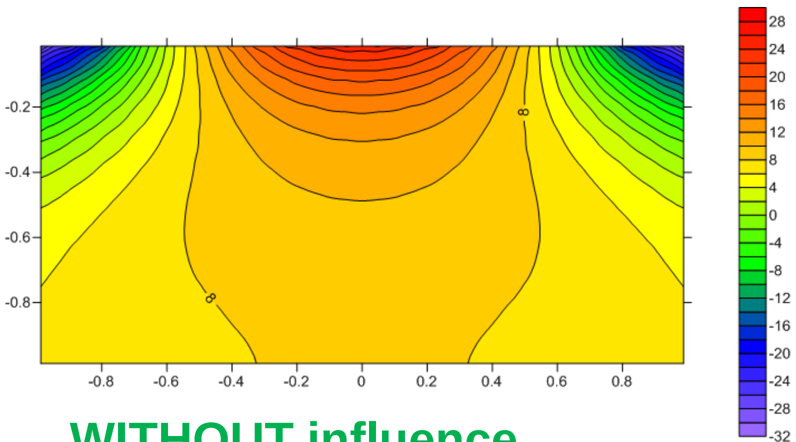


DOM solution. Output time = 1.0



<u>WITH</u> influence of deep ocean on atmosphere.

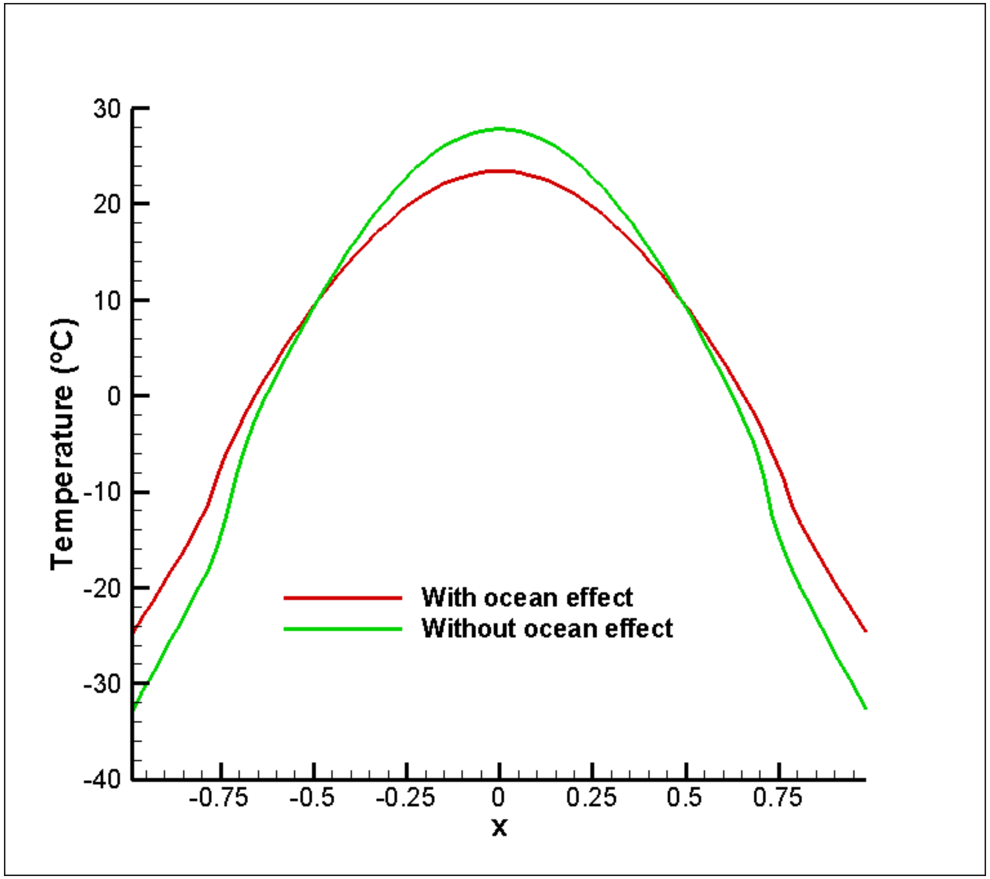
$$\frac{K_v}{D}\frac{\partial U}{\partial n} \neq 0$$



<u>WITHOUT</u> influence of deep ocean on atmosphere.

$$\frac{K_v}{D}\frac{\partial U}{\partial n} = 0$$

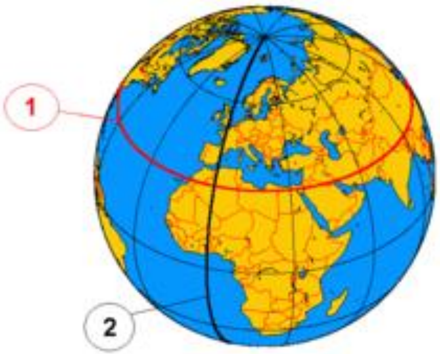
EBM solution. Output time = 1.0



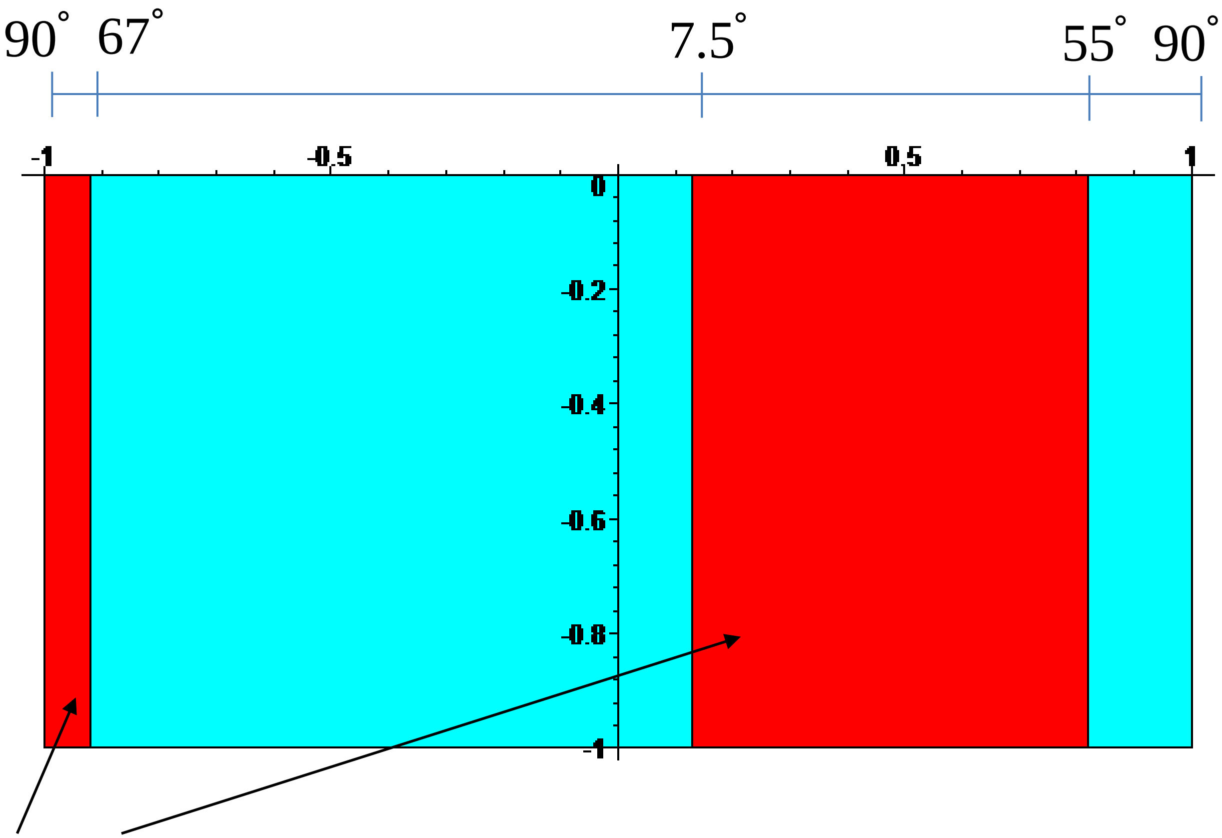
Mathematical model with land-sea distribution

$$
\left\{
\begin{aligned}
& U_t - \left(\frac{K_H}{R^2}(1-x^2)U_x\right)_x - K_V U_{zz} + \omega U_z = 0 \quad \text{in } (0,T)\times\Omega_i, \qquad i = 1 \ldots N. \\
& Du_t - \frac{DK_{H_0}}{R^2}\left((1-x^2)^{p/2}\left|u_x\right|^{p-2}u_x\right)_x + \sum_{i=1}^{M}\left(\chi_{\varpi_i} K_V \frac{\partial U}{\partial z}\right) + wxu_x + Bu + C = \\
& \qquad = \frac{1}{\rho c} QS(x)\beta(x,u) \quad \text{in } (0,T)\times\Gamma_0, \\
& wxU_x + K_V U_z = 0 \quad \text{in } (0,T)\times\Gamma_H^i, \qquad i = 1 \ldots N. \\
& (1-x^2)^{p/2}\left|u_x\right|^{p-2}u_x = 0 \quad \text{in } x = \pm 1, \\
& U_x = 0 \quad \text{in } ((0,T)\times\Gamma_{-1}^i)\cup((0,T)\times\Gamma_1^i), \\
& U(0,x,z) = U_0(x,z) \quad \text{in } \Omega_i, \\
& U(0,x,0) = u_0(x) \quad \text{in } \Gamma_0
\end{aligned}
\right.
$$

Final system



Greenwich Meridian

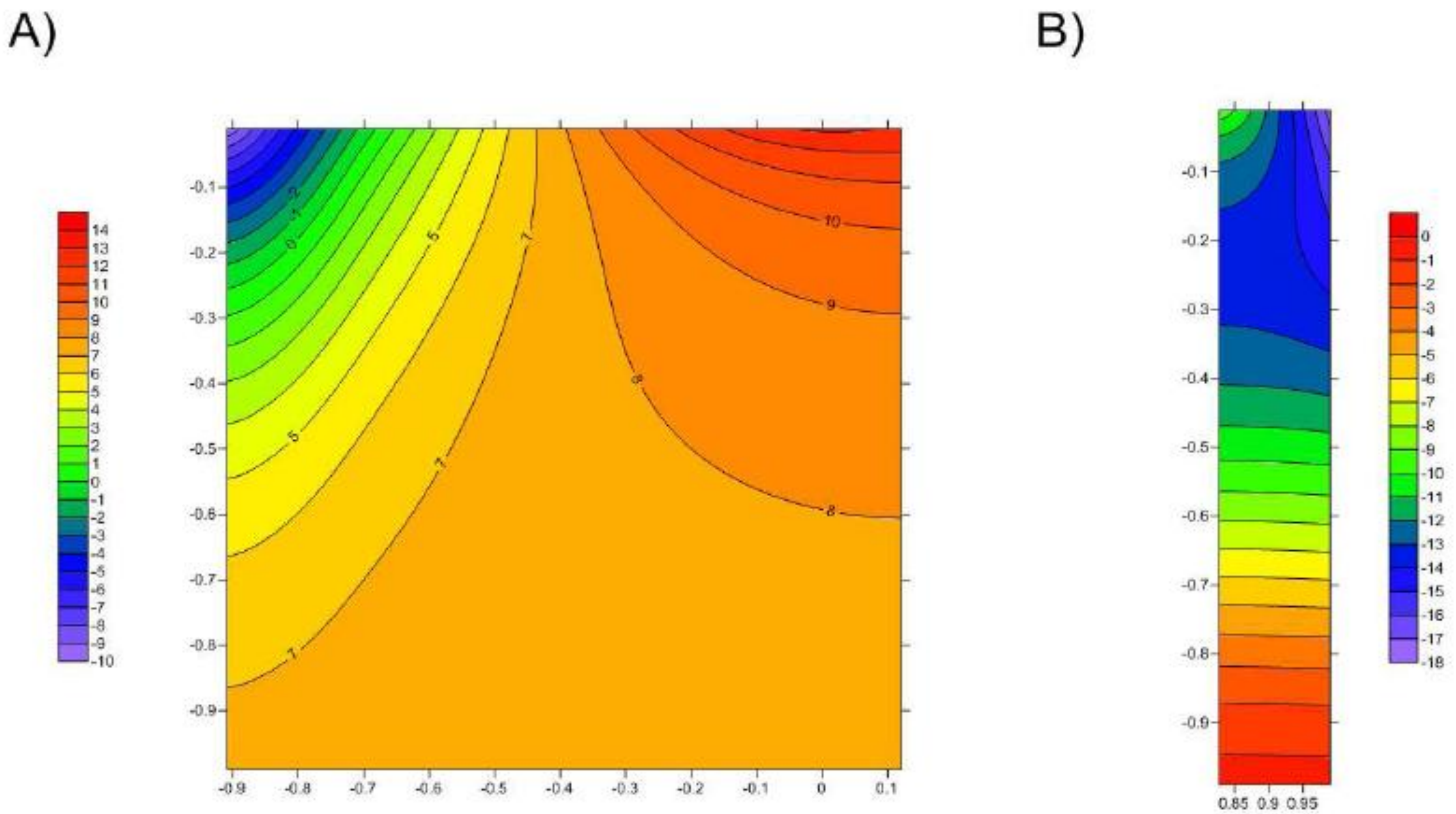


Continental zones

Mathematical model with land-sea distribution

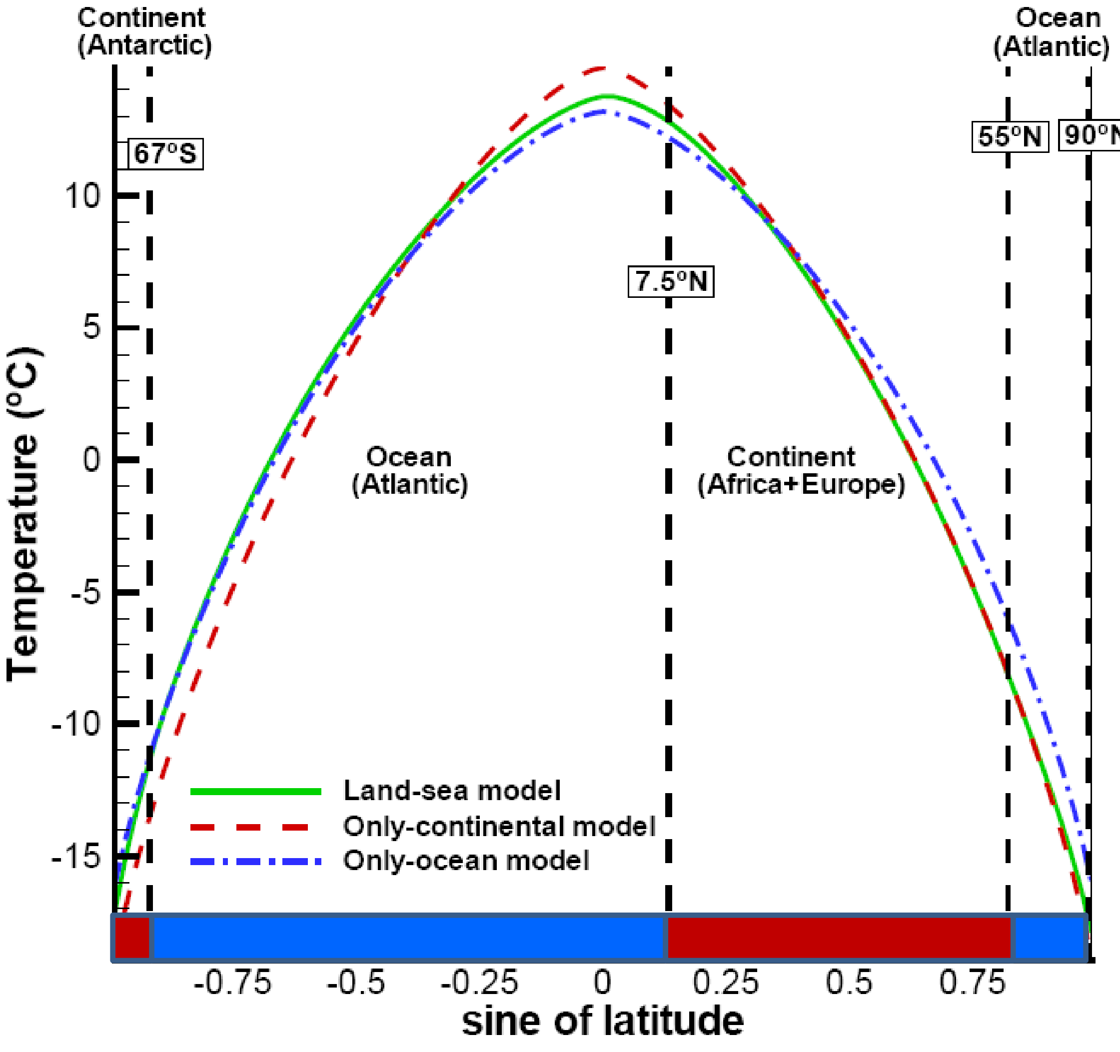
Physical parameter	$Value$
$K_H(m^2c^{-1})$	0.049
$K_{H_0}(m^2c^{-1})$	0.555×10^{-3}
$K_V(m^2c^{-1})$	0.0125
C,B	190,2
$Q(Wm^{-2})$	340
$c_w(J(kg^\circ C)^{-1})$	3900
$c_a(J(kg^\circ C)^{-1})$	1004
$\rho_w(kgm^{-3})$	1030
$\rho_a(kgm^{-3})$	1.225

Numerical results with land-sea distribution



Temperature in the deep ocean for t=5.
A) First ocean;   B) Second Ocean.

Numerical results with land-sea distribution

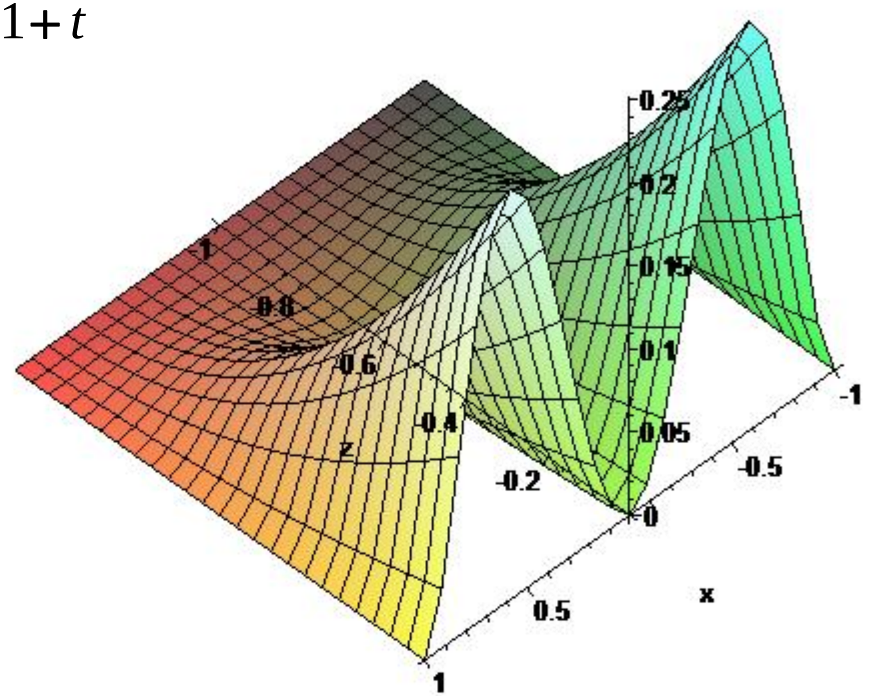


*Solution for t =5. **Full green line: land-sea model**; **dotted red line: only Continental model**; **dash-dotted blue line: Only Ocean model***

Validation of the numerical scheme with land-sea distribution

Manufactured solution:

$$U(t,x,z) = 10\frac{\left(x^2-1\right)^2 x^2 \left(1+z\right)^2}{1+t}$$



Validation of the numerical scheme with land-sea distribution

$$U_t - \left(\frac{K_H}{R^2}(1-x^2)U_x\right)_x - K_V U_{zz} + \omega U_z = \Phi(t,x,z), \quad \text{in } (0,T)\times\Omega_i, \qquad i=1,2$$

$$wxU_x + K_V U_z = 0 \quad \text{in } (0,T)\times\Gamma_H^i, \qquad i=1,2$$

$$Du_t - \frac{DK_{H_0}}{R^2}\left((1-x^2)^{3/2}\left|u_x\right|u_x\right)_x + \sum_{i=1}^{2}\left(\chi_{\varpi_i} K_V \frac{\partial U}{\partial z}\right) + wxU_x + C + Bu =$$

$$= \frac{1}{\rho c} QS(x)\beta(x,u) + \psi(t,x),$$

$$(1-x^2)^{3/2}\left|u_x\right|u_x = 0 \quad \text{in } x = \pm 1,$$

$$U_x = 0 \quad \text{in } ((0,T)\times\Gamma_{-1}^i)\cup((0,T)\times\Gamma_1^i),$$

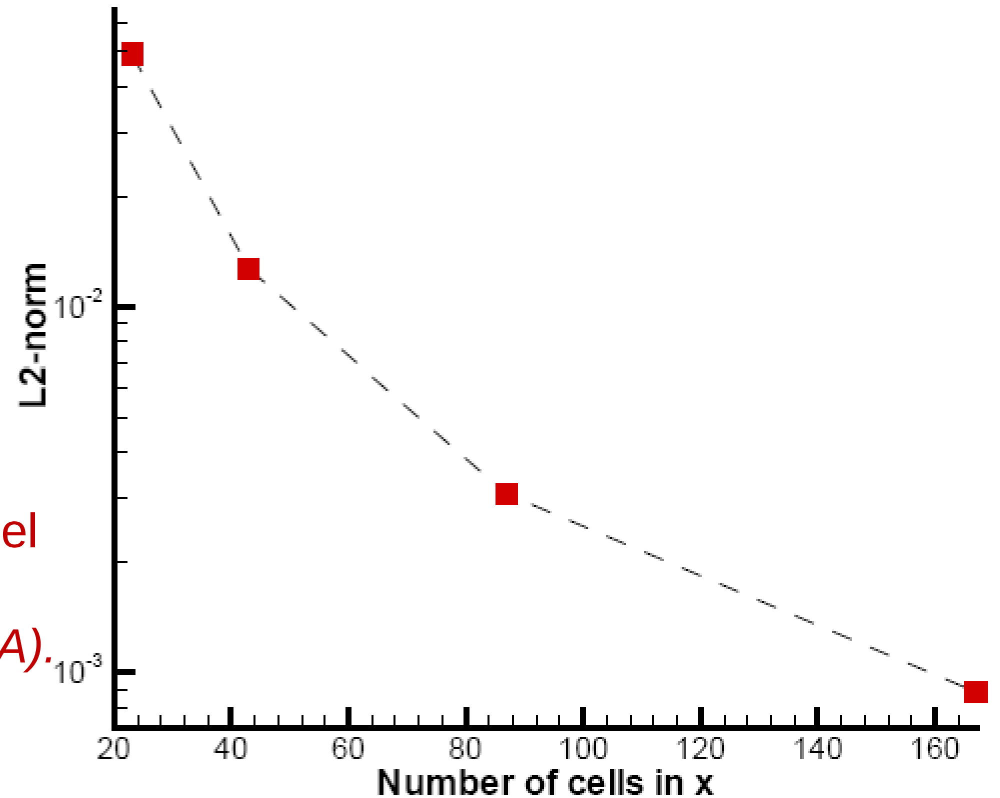
$$U(0,x,z) = 200\left(x^2-1\right)^2\left(x+0.8\right)^2\left(x+0.4\right)^2\left(x-0.4\right)^2\left(x-1\right)^2\left(1+z\right)^2,$$

$$u(0,x) = 200\left(x^2-1\right)^2\left(x+0.8\right)^2\left(x+0.4\right)^2\left(x-0.4\right)^2\left(x-1\right)^2 \quad \text{in } \Gamma_0.$$

Validation of the numerical scheme with land-sea distribution

Number of cells in x direction	L_2-norm	Order
23	4.93×10^{-2}	
43	1.27×10^{-2}	1.95
87	3.08×10^{-3}	1.98
167	8.84×10^{-4}	1.90

$$\|\epsilon\|_{L_2} = \sqrt{\Delta x \sum_{i=1}^{N_x} (u_i(t^n) - \tilde{u}_i(t^n))^2}$$



On a Climatological energy balance model with continents distribution. *Discrete and Continuous Dynamical Systems (DCDS-A). April 35 (4). doi: 10.3934/dcds.2015.35.1503 pp.1503-1519.*

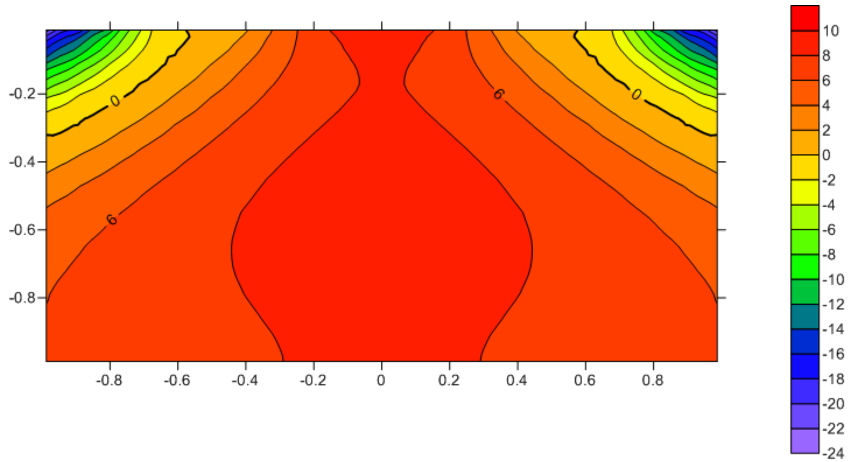
Other results (latent heat)

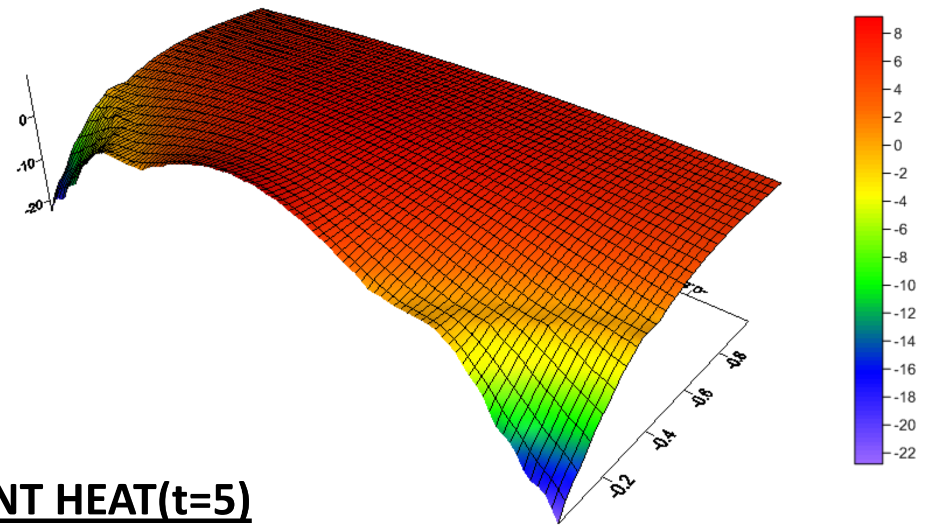
Multiple solutions and numerical analysis to the dynamic and stationary models coupling a delayed energy balance model involving latent heat and discontinuous albedo with a deep ocean

J. I. Díaz, A. Hidalgo, L. Tello

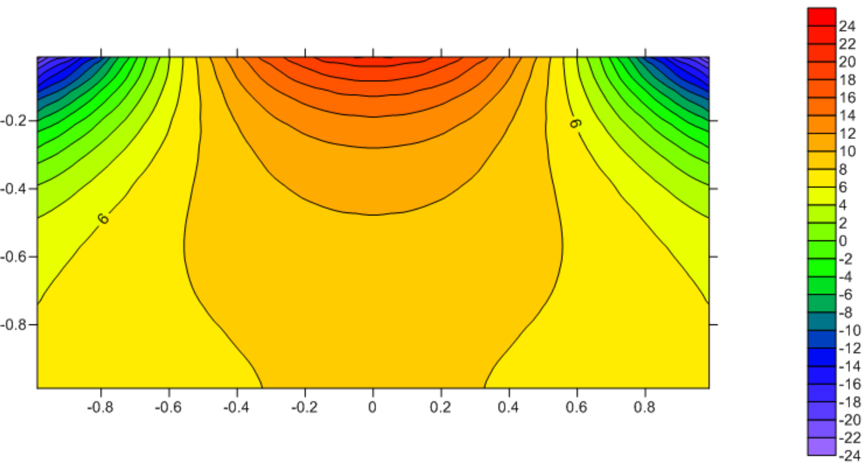
Proc. R. Soc. A: 2014 470 20140376; DOI: 10.1098/rspa.2014.0376. Published 27 August 2014

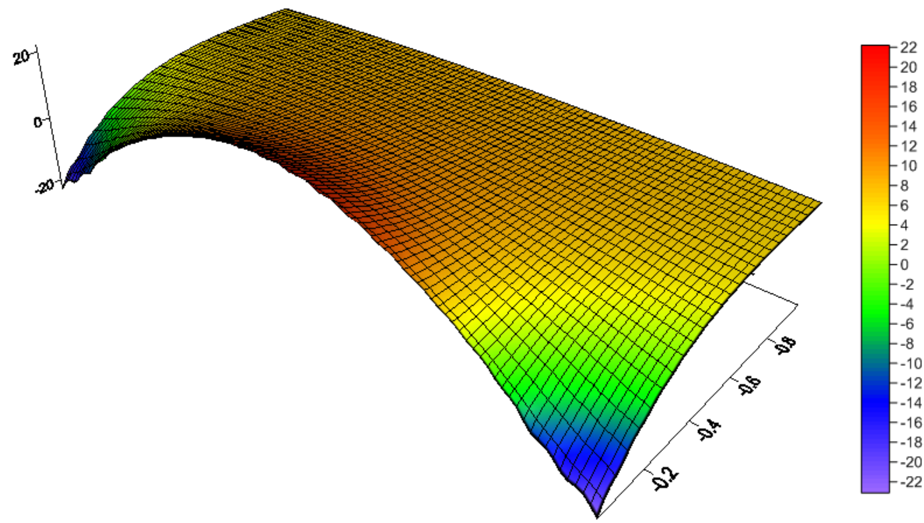
WITH LATENT HEAT (t=5)



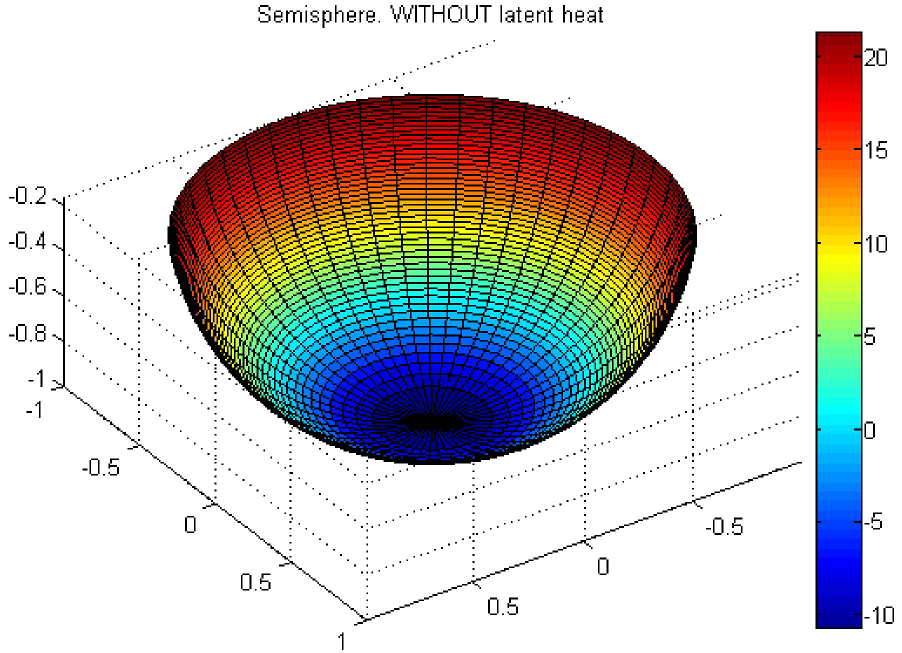


WITHOUT LATENT HEAT(t=5)

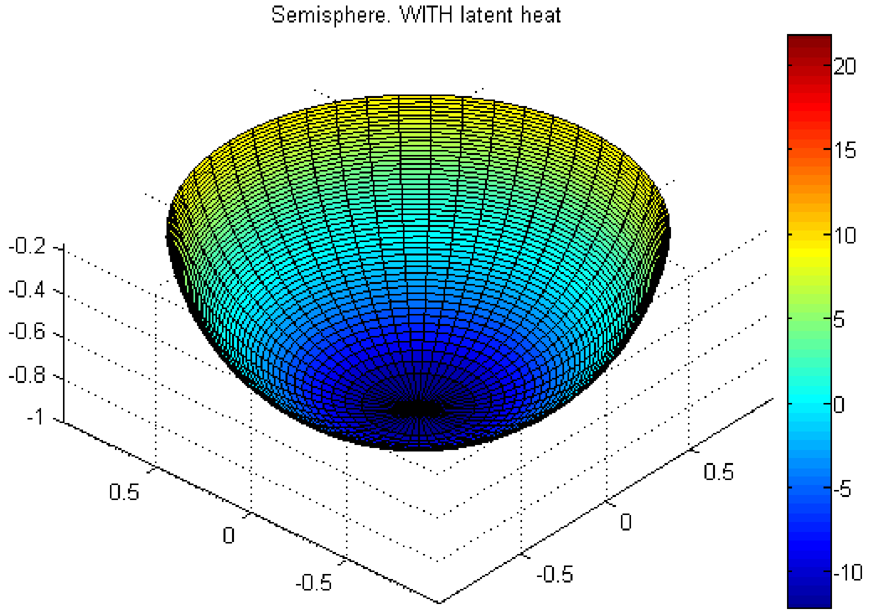


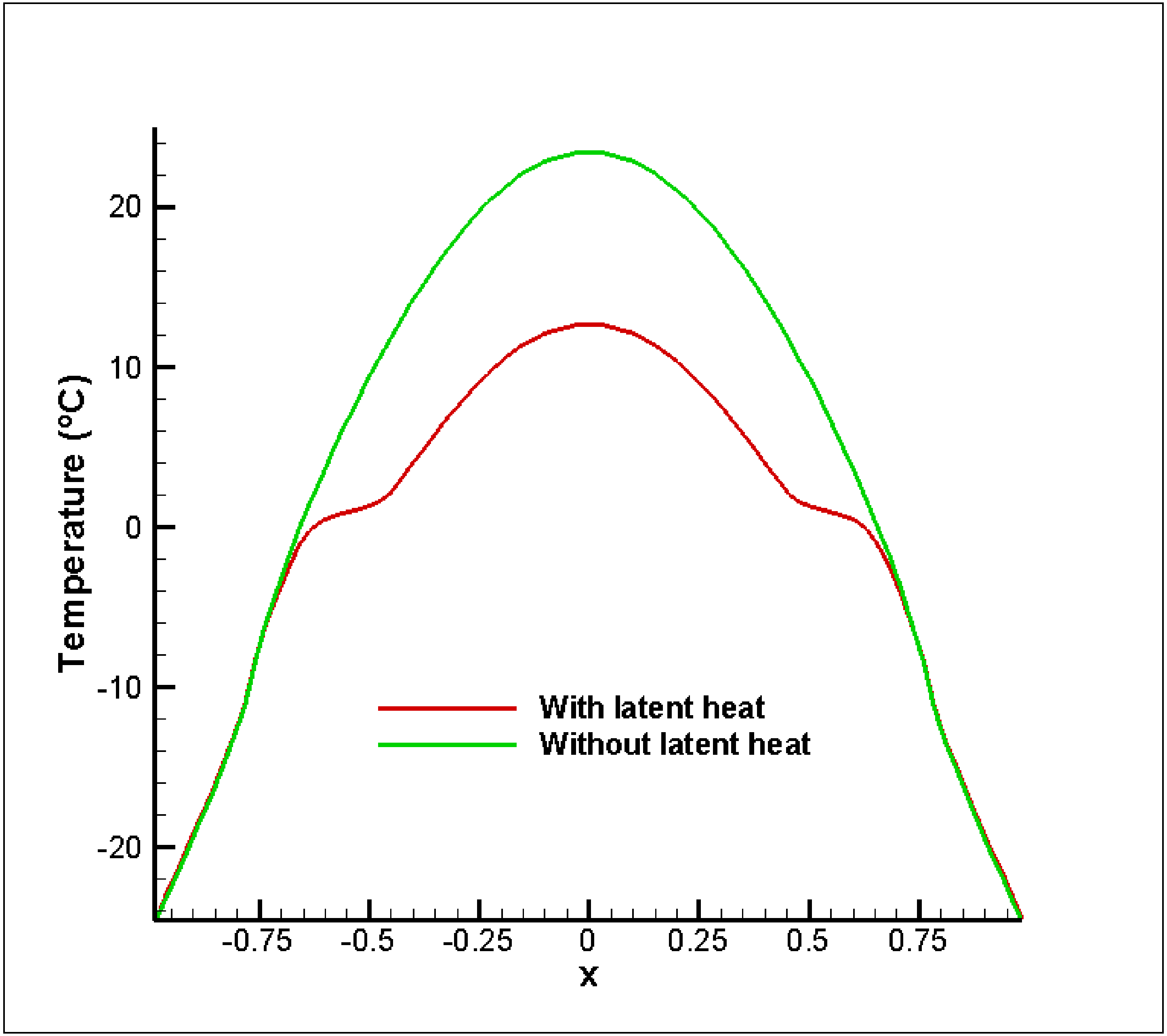


WITHOUT



WITH




20
10
0
-10
-20
Temperature (°C)
-0.75
-0.5
-0.25
0
0.25
0.5
0.75
x
With latent heat
Without latent heat

Conclusions and further research

•We have obtained the numerical solution of a 1D energy balance model with nonlinear diffusion, coupled with a 2D deep ocean model in a rectangular domain.
•The method used is a finite volume method with 3rd order Runge-Kutta TVD.
•It has been obtained the evolution of the temperature in the deep ocean and also in the surface, due to the combination of melting ice, heating-cooling of the surface of the ocean.
•The results show the thermostatic effect of the ocean.
•The effect of the land-sea distribution has been considered in the problem.
•A verification of the accuracy of the scheme has been carried out solving an auxiliary problem with known analytical solution.

•More “realistic” values of parameters.
•3D extension.